University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

HINTS FOR ASSIGNED EXERCISES 1-25

1.
Consider a uniform static magnetic field

$$\vec{B} = \hat{z}\, B_0 \ ,$$

where B_0 is a constant.
(a.)
Show that $\vec{B}$ can arise from the vector potential

$$\vec{A}_a = -B_0 y\, \hat{x} \ .$$

Hint:
In cartesian coordinates, take the curl of $\vec{A}_a$.
(b.)
Show that $\vec{B}$ can arise from the vector potential

$$\vec{A}_b = \tfrac{1}{2} B_0 s\, \hat{\phi}$$

(s and ϕ are cylindrical coordinates).
Hint:
In cylindrical coordinates, take the curl of $\vec{A}_b$. Note that s is the perpendicular distance from the point of interest to the z axis.
(c.)
By coordinate-system-independent vector analysis, show that $\vec{B}$ can arise from the vector potential

$$\vec{A} = \tfrac{1}{2} \vec{B} \times \vec{r}$$

(remember that $\vec{B}$ is constant).
Hint:
Use Identity #8 on Inside Cover (IC) #2 of Griffiths (G). Does $\vec{B}$ have any nonzero derivatives? It's easy to evaluate $\nabla \cdot \vec{r}$ and $(\vec{B} \cdot \nabla)\vec{r}$ in cartesian coordinates. Do these (trivial) results depend on the coordinate system chosen?
(d.)
Referring to G Eq. (10.7), find the gauge function λ that accomplishes the gauge transformation from $\vec{A}_a$ to $\vec{A}_b$.
Hint:
If you set $\vec{A}_a$ equal to G's $\vec{A}$, and $\vec{A}_b$ equal to his $\vec{A}'$, G Eq. (10.7) becomes

$$\vec{A}_b - \vec{A}_a = \nabla\lambda \ .$$

You'll find it convenient to express both $\vec{A}_a$ and $\vec{A}_b$ in the same coordinate system – probably cartesian is easier. (Remember that $\hat{\phi}$ is not a constant – it depends on where you are. See GIC #4.) Integrate

$$\int_0^{\vec{r}} (\nabla\lambda) \cdot d\vec{\ell}$$

and use the Gradient Theorem (GIC #2) to isolate $\lambda(\vec{r}) - \lambda(0)$; to evaluate this difference, similarly integrate the left-hand side (LHS) over any path you like.

2.
Griffiths Problem 10.3.
Hint:
The fields come from G Eq. (10.2) and (10.3) (spherical polar coordinates are convenient). As usual, get the charge and current distributions from the sourceful Maxwell equations (G Eq. (2.16) and (7.36)).

3.
Griffiths Problem 10.5.
Hint:
Use G Eq. (10.7) to do the transformation. The result should be *very* familiar!

4.
In free space with $\rho = 0$ and $\vec{J} = 0$, show that all four Maxwell equations can be obtained correctly if the scalar potential V is assumed to vanish, while the vector potential $\vec{A}$ satisfies

$$0 = \nabla \cdot \vec{A}$$
$$0 = \big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\big)\vec{A} \ .$$

Hint:
What do G Eqs. (10.2) and (10.3) tell you about the sourceless Maxwell equations for any $\vec{A}$ and

V? So let's consider the sourceful Maxwell equations. To show that $\nabla \cdot \vec{E} = 0$ as required by Gauss's law, use Eq. (10.2) to express $\vec{E}$ in terms of $\vec{A}$; then use the first of the above equations. To show that $c^2 \nabla \times \vec{B} = \partial \vec{E}/\partial t$ as required by Maxwell's version of Ampère's law, use Eq. (10.3) to express $\vec{B}$ in terms of $\vec{A}$, then apply Identity #11 (GIC #2). Use the first of the above equations to eliminate one term and the second plus Eq. (10.3) to reexpress what remains in terms of $\vec{E}$.

5.
Griffiths Problem 10.7.
Hint:
We worked this problem in class on 23 Jan – please consult your notes. We assumed that

$$\nabla \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial V}{\partial t} = s(\vec{r}, t) ,$$

where s is any nonzero time-dependent scalar field. What we wanted was

$$\nabla \cdot \vec{A}' + \epsilon_0 \mu_0 \frac{\partial V'}{\partial t} = 0$$

for a different (primed) set of potentials related to the first set by G Eq. (10.7). Using Eq. (10.7) to substitute primed potentials for the unprimed potentials in the first equation above, and taking advantage of a cancellation from the second equation, we were left with a result of the form G (10.14) with λ replacing V and s replacing ρ/ϵ_0. If we know how to solve (10.14) for V, do we know how to solve the equation you got for λ?

6.
Consider the Levi-Civita density $\epsilon_{ijk} \equiv 1$ (ijk = even permutation of 123); $\equiv -1$ (odd permutation of 123); $\equiv 0$ (otherwise). It is found, for example, in the cross product

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k .$$

Note that summation over the repeated indices j and k is implied; their domain is $1 \leq j, k \leq 3$.
(a.)
Show that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} ,$$

where δ is the Kronecker delta function (whose elements are those of the unit matrix).
Hint:
ϵ_{ijk} and ϵ_{klm} vanish unless $i \neq j \neq k$ and $k \neq l \neq m$. Therefore only one value of k yields a nonzero term, which occurs either if $i = l$ and $j = m$ or if $i = m$ and $j = l$. Suppose that k takes on its one useful value, that $i = l$ and $j = m$, and that ijk is an even permutation (even number of adjacent swaps) of 123. What value does $\epsilon_{ijk}\epsilon_{klm}$ take under those circumstances? Work out the other three possibilities.
(b.)
The determinant of a 3×3 matrix is given by

$$\det A \propto \epsilon_{ijk} A_{il} A_{jm} A_{kn} \epsilon_{lmn} .$$

By considering the number of nonzero terms on the RHS, and comparing it to the number of terms you would have expected for a 3×3 determinant, deduce the constant of proportionality. Express it in terms of a factorial.
Hint (due to Aaron Chen):
Consider the special case $A_{ij} = \delta_{ij}$, *i.e.* A is the unit matrix. Then $\det A = \epsilon_{ijk}\epsilon_{ijk}$. An individual term in this sum is equal to 1 if $i \neq j \neq k$ and 0 otherwise. How many different combinations of i, j, and k are there in which i, j, and k are all different?
(c.)
Guessing the explicit constant of proportionality, write a similar equation for the determinant of a 4×4 matrix. How should ϵ_{ijkl} be defined?
Hint:
How many different combinations of i, j, k, and l are there in which i, j, k, and l are all different? In defining ϵ_{ijkl}, consider the fact that swapping two adjacent indices of ϵ_{ijkl} is like swapping two adjacent rows of the matrix whose determinant is calculated; swapping two adjacent indices of ϵ_{mnop} is like swapping two adjacent columns of that matrix.

7.
Griffiths Problem 12.55. Don't get fooled by the typo – he means "$\partial^\mu \equiv \partial/\partial x_\mu$".
Hint:
Consider the *direct* Lorentz transformation of the spacetime four-vector x:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \ .$$

Now, for any linear transformation

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu \ .$$

Therefore

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \ .$$

Likewise, considering the *inverse* transformation of x, show that

$$(\Lambda^{-1})^\nu{}_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \ .$$

Finally, use the chain rule to express $\partial/\partial x'^\mu$ in terms of $\partial x^\nu/\partial x'^\mu$ and $\partial/\partial x^\nu$.

8.
An object a^μ is a (contravariant) four-vector if it transforms (between frames as defined in Short Course in Special Relativity (SCSR) Fig. 2) according to

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu \ ,$$

where Λ is the (symmetric) 4×4 Lorentz transformation matrix. (Conventionally, the first (superscript) index labels the row and the second (subscript) index labels the column, but this makes no difference for a symmetric matrix.) Covariant four-vectors instead transform according to

$$a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu$$

(otherwise the scalar product $a_\mu a^\mu = a'_\mu a'^\mu$ would not remain invariant for different Lorentz frames). Consider now an (arbitrary) four-tensor $H^{\mu\nu}$. In frame $\mathcal{S}$, $H^{\mu\nu}$ contracts with covariant four-vector a_ν to yield contravariant four-vector b^μ, according to

$$b^\mu = H^{\mu\nu} a_\nu \ .$$

In the frame $\mathcal{S}'$, requiring $H^{\mu\nu}$ to satisfy the transformation properties of a four-tensor, we define $H'^{\mu\nu}$ so that

$$b'^\mu = H'^{\mu\nu} a'_\nu \ .$$

Prove that

$$H'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma H^{\rho\sigma} \ .$$

This defines the Lorentz transformation property of a four-tensor.
Hint:
Start from

$$b^\rho = H^{\rho\sigma} a_\sigma \ .$$

Substitute

$$b^\rho = (\Lambda^{-1})^\rho{}_\alpha b'^\alpha$$

and

$$a_\sigma = a'_\nu \Lambda^\nu{}_\sigma \ .$$

Then multiply both sides of the resulting equation by $\Lambda^\mu{}_\rho$ and use the fact that

$$\Lambda^\mu{}_\rho (\Lambda^{-1})^\rho{}_\alpha = \delta^\mu{}_\alpha \ ,$$

where $\delta^\mu{}_\alpha$ is an element of the unit matrix. Use this fact to simplify the left-hand side. Then compare your equation to

$$b'^\mu = H'^{\mu\nu} a'_\nu$$

and draw the desired conclusion.

9.
Consider the antisymmetric *electromagnetic field strength tensor*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ ,$$

where both ∂^μ and A^μ are (contravariant) four-vectors. Prove that $F^{\mu\nu}$ is a four-tensor, *i.e.* it transforms according to the result of Problem 8.
Hint:
Start from

$$F'^{\mu\nu} = \partial'^\mu A'^\nu - \partial'^\nu A'^\mu \ .$$

Write

$$\partial'^\mu = \Lambda^\mu{}_\rho \partial^\rho$$
$$A'^\nu = \Lambda^\nu{}_\sigma A^\sigma$$

(similarly for ∂'^ν and A'^μ), plug in, and collect terms.

10. (Light cone)
(a.)
Event A occurs at spacetime point $(ct, x, y, z) = (0, 1, 1, 1)$; event B occurs at $(1, 0, 0, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events A and B occur at the same spatial coordinates? If so, find $c|t'_A - t'_B|$, c times the magnitude of the time interval in $\mathcal{S}'$ between the two events.
Hint:
Denote by x_A (x_B) the spacetime coordinate of event A (B); $\Delta x \equiv x_B - x_A$. $\Delta x \cdot \Delta x$ is the same in any Lorentz frame. Is Δx timelike? If so, how is $\Delta x' \cdot \Delta x'$ related to $c|t'_A - t'_B|$?
(b.)
Is there an inertial system $\mathcal{S}''$ in which events A and B occur simultaneously? If so, find $|\vec{r}''_A - \vec{r}''_B|$, the distance in $\mathcal{S}''$ between the two events.
Hint:
Is Δx spacelike? If so, how is $\Delta x'' \cdot \Delta x''$ related to $|\vec{r}''_A - \vec{r}''_B|$?
(c.)
Can event A be the cause of event B, or vice versa? Explain.
Hint:
Is Δx timelike?
(d.)
Event D occurs at spacetime point $(ct, x, y, z) = (-1, 0, 0, 0)$; event E occurs at $(2, 1, 1, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events D and E occur simultaneously? If so, find $|\vec{r}'_E - \vec{r}'_D|$, the magnitude of the distance in $\mathcal{S}'$ between the two events.
Hint:
Is Δx spacelike? If so, how is $\Delta x' \cdot \Delta x'$ related to $|\vec{r}'_E - \vec{r}'_D|$?
(e.)
Is there an inertial system $\mathcal{S}''$ in which events D and E occur at the same spatial coordinates? If so, find $c|t''_E - t''_D|$, c times the magnitude of the time interval in $\mathcal{S}''$ between the two events.
Hint:
Is Δx timelike? If so, how is $\Delta x'' \cdot \Delta x''$ related to $c|t''_E - t''_D|$?

11.
Using *e.g.* the method of Short Course in Special Relativity [SCSR] §7, obtain the 4×4 Lorentz transformation matrix for the case in which frame $\mathcal{S}'$ moves with respect to frame $\mathcal{S}$ with speed $\beta_0 c$ in an *arbitrary* direction $(n_1, n_2, 0)$ in the x-y plane, where $\vec{n}$ is a unit vector.
Hint:
Consult SCSR §7.

12.
(a.)
In SCSR §8, clock time intervals measured in a frame in which the clock is not at rest are shown to be *dilated* by the factor γ_0. This analysis used the *inverse* Lorentz transformation. Reanalyze the same problem using the *direct* Lorentz transformation. Is the answer the same?
Hint:
The *temporal* direct Lorentz transformation is

$$\begin{aligned} ct'_2 &= \gamma_0 ct_2 - \gamma_0 \beta_0 x_2 \\ ct'_1 &= \gamma_0 ct_1 - \gamma_0 \beta_0 x_1 \ . \end{aligned}$$

Subtracting,

$$\gamma_0 c\Delta t = c\Delta t' + \gamma_0 \beta_0 \Delta x \ .$$

In order to eliminate Δx, apply the *spatial* direct Lorentz transformation and use the fact that $\Delta x' = 0$.
(b.)
In SCSR §9, the length of a rod measured in a frame in which the rod is not at rest is shown to be *contracted* by the factor $1/\gamma_0$. This analysis used the *direct* Lorentz transformation. Reanalyze the same problem using the *inverse* Lorentz transformation. Is the answer the same?
Hint:
The *spatial* inverse Lorentz transformation is

$$\begin{aligned} x_2 &= \gamma_0 x'_2 + \gamma_0 \beta_0 ct'_2 \\ x_1 &= \gamma_0 x'_1 + \gamma_0 \beta_0 ct'_1 \ . \end{aligned}$$

Subtracting,

$$\Delta x = \gamma_0 \Delta x' + \gamma_0 \beta_0 c\Delta t' \ .$$

In order to eliminate $\Delta t'$, apply the *temporal* inverse Lorentz transformation and use the fact that $\Delta t = 0$.

13. (Addition of velocities)
In texts that do not emphasize the rapidity or boost parameter η, the Einstein law for the addition of velocities is derived less elegantly as follows (see SCSR Fig. 7). Denote by x^1 (x'^1) the x coordinate of the origin of $\mathcal{S}''$ as observed in the lab frame $\mathcal{S}$ (moving frame $\mathcal{S}'$). Write a standard inverse Lorentz transformation

$$x^0 = \gamma x'^0 + \gamma\beta x'^1$$
$$x^1 = \gamma x'^1 + \gamma\beta x'^0 \ .$$

Then take the differential of it: $dx^0 = \ldots$; $dx^1 = \ldots$. Divide the bottom by the top equation and identify

$$\frac{dx^1}{dx^0} = \beta'' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S}$$
$$\frac{dx'^1}{dx'^0} = \beta' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S}' \ .$$

Obtain the Einstein law for the addition of velocities (SCSR Eq. (24)):

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'} \ .$$

Hints are already embedded in the statement of this problem.

14.
Consider the standard case in which two Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$ coincide at $t = t' = 0$, with frame $\mathcal{S}'$ moving at velocity $\beta c\,\hat{x}$ with respect to frame $\mathcal{S}$. As seen in a third frame $\mathcal{S}''$, also moving along $\hat{x}$ with respect to $\mathcal{S}$, two clocks fixed to the origins of frames $\mathcal{S}$ and $\mathcal{S}'$, respectively, appear to agree. With respect to frame $\mathcal{S}$, considering that *rapidity ("boost") is the additive parameter of the Lorentz transformation*, show that the speed $\beta''c$ of frame $\mathcal{S}''$ is given by

$$\beta'' = \tanh\!\left(\tfrac{1}{2}\tanh^{-1}\beta\right) \ .$$

Hint:
If the clocks in $\mathcal{S}$ and $\mathcal{S}'$ agree, what does that say about the |boost| of each frame relative to frame $\mathcal{S}''$? If frames $\mathcal{S}$ and $\mathcal{S}'$ are different, can the signs of their boosts relative to frame $\mathcal{S}''$ be the same? Keeping in mind that rapidity (boost) is the additive parameter of the Lorentz transformation, write a simple equation describing how the boosts η'' and η are related. Then convert that equation to an equation in β'' and β.

15. (Taylor & Wheeler problem 51)
The clock paradox, version 3.
Hints are already embedded in the statement of this problem, which is lengthy and not repeated here.

16. (Surface muons)
"Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays to an (anti)muon (μ^+, a heavy electron-like particle) and a muon neutrino (ν_μ) via

$$\pi^+ \to \mu^+ + \nu_\mu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have $\frac{3}{4}$ of the rest mass of a pion; neglect the neutrino mass.
(a.)
Show that the surface muons travel at a speed which is a fraction $\beta_0 = \frac{7}{25}$ of the speed of light.
Hint:
Take π, μ, and ν to be the four-momenta of the pion, muon, and neutrino, respectively. Express energy-momentum conservation two ways:

$$\pi - \mu = \nu$$
$$\pi - \nu = \mu \ .$$

For each equation take the dot product of the LHS and RHS with itself. Solve the first equation for E_μ and the second for E_ν. Given that the neutrino is massless, what can you say about the relationship between E_ν and $c|\vec{p}_\nu|$? Given that momentum is conserved, what can you say about the relationship between $\vec{p}_\nu$ and $\vec{p}_\mu$? How is β_μ related to E_μ and $c|\vec{p}_\mu|$?

(b.)

If a muon's mean proper lifetime is τ, what fraction of the muons will decay during a flight path of length L in the laboratory? Express your answer in terms of β_0.

Hint:

Apply time dilation to find the muon's mean lab lifetime, and use the muon's lab velocity to convert its mean lab lifetime to a mean lab path length. For spontaneous decay, remember that the survival probability depends exponentially on the appropriate parameter (time or distance).

17.

In the lab frame $\mathcal{S}$, a particle with velocity βc $= \frac{4}{5}c$ decays into two massless particles with the same energy each.

(a.)

If the parent particle has mean (proper) lifetime τ in its *own* rest frame $\mathcal{S}'$, calculate its mean flight path L in the lab frame $\mathcal{S}$.

Hint:

Apply time dilation to find the parent particle's mean lab lifetime, and use the particle's lab velocity to convert its mean lab lifetime to a mean lab flight path.

(b.)

In the lab frame $\mathcal{S}$, calculate the opening angle $\psi = \cos^{-1} \hat{p}_1 \cdot \hat{p}_2$ between the two daughter particles.

Hint:

Take P, p_1, and p_2 to be the four-momenta of the parent and two daughters. Expressing energy-momentum conservation as $P = p_1 + p_2$, take the dot product of the LHS and RHS with itself. Given that the daughter particles are massless, how are $c|\vec{p}_{1,2}|$ related to $E_{1,2}$? Given that energy is conserved, how are $E_{1,2}$ related to the parent energy E? Given the parent's speed βc, how is $c|\vec{P}|$ related to E? Solve for $\cos \hat{p}_1 \cdot \hat{p}_2$.

18.

Here's an adult version of Griffiths' Problem 12.35. In a pair annihilation experiment, a positron (mass m) with total energy $E = \gamma mc^2$ hits an electron (same mass, but opposite charge) at rest. (Griffiths has it the other way around, but that's unrealistic – it's easy to make a positron beam, but hard to make a positron target.) The two particles annihilate, producing two photons. (If only one photon were produced, energy-momentum conservation would force it to be a massive particle traveling at a velocity less than c.) If one of the photons emerges at angle θ relative to the incident positron direction, show that its energy ϵ is given by

$$\frac{mc^2}{\epsilon} = 1 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} \cos\theta \ .$$

(In particular, if the photon emerges perpendicular to the beam, its energy is equal to mc^2, independent of the beam energy. Similar results have been used to design clever experiments.)

Hint:

Take a, b, d, and e to be the four-momenta in this 2+2 reaction $a + b \to d + e$; note that particles a and b have mass m, that particle b is at rest, and that particle d emerges at angle θ with respect to the direction of a. Because the least is known about particle e, express energy-momentum conservation as

$$a + b - d = e$$

and take the dot product of the LHS and RHS with itself. Your expression should involve E_a, $\vec{p}_a$, and $\vec{p}_d$ as well as m and ϵ (the energy of d). Use the fact that particle d is massless to express $c|\vec{p}_d|$ in terms of ϵ. You are given the Lorentz factor γ of particle a. Eliminate E_a by expressing it in terms of m and γ, and eliminate $|\vec{p}_a|$ by expressing it in terms of m, γ, and $\beta(\gamma)$.

19.

If you have studied Rutherford scattering (elastic scattering of a nonrelativistic He nucleus from an Au nucleus), you have seen the *differential cross section* for this process written in the form

$$\frac{d\sigma}{d\Omega} \propto \frac{(zZe^2)^2}{\sin^4 \frac{\Theta}{2}} \ ,$$

where ze (Ze) is the electric charge of the He (Au) nucleus; Θ is the angle by which the He nucleus is elastically scattered, measured in the CM frame; and $d\Omega$ is an element of solid angle within which the He nucleus emerges in that frame. [A century ago, Rutherford-scattering data collected by graduate students who were used as particle detectors demonstrated that atoms contain charged point-like constituents (nuclei).] Here we revisit Rutherford scattering for *relativistic* particles.

Consider the 2+2 relativistic scattering process

$$p + a \rightarrow q + b\,,$$

where p, a, q, and b denote both the particles and their 4-momenta. The *Mandelstam variables*, first written down by Berkeley emeritus professor Stanley Mandelstam, are

$$\begin{aligned} s &\equiv (p+a)\cdot(p+a) \equiv \text{CM energy}^2 \\ t &\equiv (q-p)\cdot(q-p) \equiv \text{4 momentum transfer}^2 \\ u &\equiv (b-p)\cdot(b-p) \equiv \text{cross channel transfer}^2 \end{aligned}$$

In this problem we are concerned with the Mandelstam variable t.

(a.)
Further assuming that the masses of particles p and q are negligible, show that

$$-t = 4\tfrac{E}{c}\tfrac{E'}{c}\sin^2\tfrac{\Theta}{2}\,,$$

where E (E') is the energy of particle p (q), and Θ is the angle between $\vec{p}$ and $\vec{q}$.
Hint:
Take the dot product of $(q-p)$ with itself; use the fact that particles q and p are massless to express $c|\vec{p}|$ and $c|\vec{q}|$ in terms of E and E'.
(b.)
Why is $d(-t)$ a Lorentz invariant? $d\sigma$ is an area transverse to the beam direction. Why is $d\sigma$ invariant to Lorentz transformations along that direction? In a system of units where $\hbar = c = 1$, all quantities have dimensions that can be expressed in units of Joules. In those units, what are the dimensions of $d\sigma$?
Hint:
The constant $\hbar c$ has dimensions of (energy) $\times$ (distance) [$\hbar c \approx 197 \times 10^{-9}$ eV m in SI]. However, in this problem's system of units $\hbar c$ is equal to unity, allowing you to multiply or divide by it at will.
(c.)
In part (b.) you showed that $d\sigma/d(-t)$ is a Lorentz invariant. If particle p (which becomes particle q) and particle a (which becomes particle b) both are structureless, and if the scattering is elastic (particles a and b both have the same mass), the only relevant Lorentz-invariant variable that is available to us is $-t$. On purely dimensional grounds, show that

$$\frac{d\sigma}{d(-t)} \propto \frac{1}{t^2}\,.$$

[If p (a) has electric charge ze (Ze), and they interact electromagnetically, the constant of proportionality is $4\pi z^2 Z^2 \alpha^2$, where the fine structure constant α is given as usual by $4\pi\epsilon_0\alpha = e^2/\hbar c$. This formula is correct to the extent that Z or $z \times (\alpha \approx 1/137)$ can be neglected relative to unity.]
(d.)
Under all of these conditions, using the results of (a.) and (c.) and working in the center of mass, show that the nonrelativistic elastic scattering result

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\Theta}{2}}$$

does remain valid even when relativistic effects are taken into account.
Hint:
In the CM, the elastic scattering process is specified completely by the single variable Θ. Express $-t$ and $d(-t)$ in terms of Θ and $d\Theta$. Then, integrating over azimuth, express $d\Theta$ in terms of $d\Omega$.

20. (Relations used in particle physics)
(a.) Lorentz-invariant phase space [LIPS]
By transforming dp_x and E while keeping p_y and p_z fixed, show that

$$\frac{c\,dp_x}{E}dp_y dp_z$$

is invariant to a boost along x. (Since one can always define x to be the boost direction, and $d^3p \equiv dp_x dp_y dp_z$, a LIPS element

$$\frac{c\,d^3p}{E}$$

is invariant to a boost in *any* direction and therefore is Lorentz invariant.)

Hint:
You are trying to show

$$\frac{c\,dp'_x dp'_y dp'_z}{E'} = \frac{c\,dp_x dp_y dp_z}{E} \; .$$

Use a direct Lorentz transformation along x to express dp'_x in terms of dp_x and dE, and to express E' in terms of E and p_x. Consider the fact that the four-momentum^2 is equal to m^2c^2:

$$\tfrac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = m^2c^2 \; .$$

Holding p_y and p_z constant, take the differential of each side to obtain a relation between dE and dp_x. Use this relation to eliminate dE from your expression for dp'_x; a welcome cancellation should result.

(b.)
Suppose a particle has momentum $\vec{p}$ and energy E. Define the particle's *longitudinal rapidity* y to be the boost along x that would be needed to make $p'_x = 0$ in the new frame $\mathcal{S}'$. Show that

$$y = \tanh^{-1} \frac{cp_x}{E} \; .$$

If your calculator doesn't have an arc hyperbolic tangent button, use the equivalent definitions

$$y = \tfrac{1}{2} \ln \frac{E/c + p_x}{E/c - p_x}$$
$$y = \ln \frac{E/c + p_x}{\sqrt{p_y^2 + p_z^2 + m^2c^2}} \; .$$

Hint:
How is the particle's |velocity| $|\vec{\beta}|c$ related to $|\vec{p}|$ and E? How is the x component $\beta_x c$ of that velocity related to p_x and E? What relative velocity β_0 of $\mathcal{S}'$ along the x direction would cause β'_x and therefore p'_x to vanish? How is β_0 related to the required boost?

(c.)
Using the fact that the rapidity (boost) is the additive parameter for the Lorentz transformation, and that y is defined to be a boost along $\hat{x}$, argue that an increment dy in longitudinal rapidity must be the same in two Lorentz frames that differ only by a relative boost along $\hat{x}$. Use this argument to conclude that

$$dy\, dp_y dp_z$$

is invariant to boosts along $\hat{x}$, as was the LIPS element

$$\frac{c\,d^3p}{E}$$

in part (a.). (In fact, these two expressions are equal.) Invariance of the longitudinal rapidity interval dy is a godsend for proton collider users. Since the proton's interacting constituents (quarks or gluons) carry only a variable fraction of the proton momentum, the center of mass (CM) of the colliding constituents is boosted along the beam direction by a variable amount (typically of order unity). However, the *difference* in longitudinal rapidity between any pair of emitted particles is unaffected by this unwelcome CM boost.)

(d.)
Define the *pseudorapidity* y_{pseudo} as the longitudinal rapidity that a particle would have if it were ultrarelativistic. Show that

$$y_{\text{pseudo}} = \tanh^{-1}(\cos\theta) \; ,$$

where θ is the angle between the particle's direction and the x axis. (This is another godsend: if a particle is known to be ultrarelativistic, its longitudinal rapidity can be approximated by its pseudorapidity, which can be measured by knowing only the particle's direction.) An equivalent definition of pseudorapidity is

$$y_{\text{pseudo}} = -\ln\left(\tan\tfrac{\theta}{2}\right) \; .$$

Hint:
If the particle is ultrarelativistic, what is the relation between $c|\vec{p}|$ and E?

21. (Effect of inefficient rocket engine)
All of the energy put out by the rocket engine depicted in SCSR Fig. 9 consists of particles emitted straight out the back. Consider the more realistic case in which only a fraction ϵ of the energy output consists of such particles; as seen in a Lorentz frame comoving with the rocket, the balance of

the energy is emitted isotropically, for example as thermal photons. Therefore ϵ is the engine's *efficiency*. Otherwise adopt the conditions of SCSR Fig. 9 and carry out a derivation analogous to that found in SCSR §14. Assuming that it starts from rest, show that the final boost of the rocket is reduced directly by this efficiency factor:

$$\eta_{\text{final}} = \epsilon\, |\vec{\beta}_1| \ln \frac{m_0}{m_{\text{final}}} \ .$$

Hint:
For a perfectly efficient engine, the first set of equations in SCSR §14 is

$$\begin{aligned} P_0 &= (mc, \vec{0}) \\ P' &\approx \big((m - dm)c + \tfrac{1}{2} mc|d\vec{\beta}|^2, mc\, d\vec{\beta}\big) \\ p_1 &= \big(\frac{dE_1}{c}, \vec{\beta}_1 \frac{dE_1}{c}\big) \ . \end{aligned}$$

When the engine becomes inefficient, a third class of final-state elements is created: photons that carry off energy but are emitted isotropically. Define another four-momentum

$$q_1 = \big(\frac{dF_1}{c}, \vec{q}_1\big)$$

to represent the sum of these photons. What is the value of their net momentum $\vec{q}_1$? How is their total energy dF_1 related to dE_1?

22. (Wave aberration)
Please refer to SCSR Fig. 10. Consider Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with spatial origins coincident at $t = t' = 0$. As usual, frame $\mathcal{S}'$ moves in the $\hat{x} = \hat{x}'$ direction with velocity $\beta_0 c$ relative to frame $\mathcal{S}$. A wave is emitted by a source that is at rest with respect to $\mathcal{S}'$. As seen by an observer in the lab frame $\mathcal{S}$, the wave travels with phase velocity $\beta_{\text{ph}} c$ at an angle θ with respect to the $\hat{x}$ direction ($\theta = 0$ if directly approaching, $\theta = \pi$ if directly receding). However, as seen by an observer who is at rest with respect to the frame $\mathcal{S}'$, show that the wave makes a different angle θ' with respect to the $\hat{x}'$ direction, where

$$\tan\theta' = \frac{\sin\theta}{\gamma_0(\cos\theta - \beta_0\beta_{\text{ph}})} \ .$$

Hint:
Defining the y axis so that the wave travels in the xy plane, write direct Lorentz transformations for k'_y and k'_x.

23. (Lorentz transformation of EM fields)
Consider Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with frame $\mathcal{S}'$ moving in the $\hat{x} = \hat{x}'$ direction with velocity $\beta_0 c$ relative to frame $\mathcal{S}$. Using the Lorentz transformation for the field strength tensor,

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma} \ ,$$

and considering explicitly the values of the elements of $F^{\mu\nu}$, as given by SCSR Eq. (62), show that

$$E'_y = \gamma_0(E_y - \beta_0 c B_z) \ ,$$

as claimed by Griffiths' Eq. (12.102).
Hint: Evaluate F'^{02} using the elements of Λ and F.

24. (Relativistic electron-positron beams)
In a straight channel oriented along the $\hat{z}$ axis there are two opposing beams:

- a beam of positrons (charge $+e$) with velocity $+\hat{z}\beta_0 c$.
- a beam of electrons (charge $-e$) with velocity $-\hat{z}\beta_0 c$.

Each beam is confined to a small cylindrical volume of cross sectional area A centered on the $\hat{z}$ axis. Within that volume, there is a uniform number density $= n$ positrons/m^3 and n electrons/m^3.
(a.)
In terms of n, A, e, and β_0, calculate the total current I in the channel due to the sum of both beams (note $I \neq 0$).
Hint:
For a positron plasma of uniform velocity $\vec{v}$ and number density n, the current density $\vec{J} = +en\vec{v}$, and the current $I = \iint \vec{J} \cdot d\vec{a}$.
(b.)
Use Ampère's Law to calculate the (azimuthal) magnetic field $\vec{B}$ outside the channel a distance s from the $\hat{z}$ axis.

Consider now a Lorentz frame $\mathcal{S}'$ traveling in the $\hat{z}$ direction with velocity $\beta_0 c$ relative to the lab frame described above. (This β_0 is the same β_0 as above.)

(c.)

As seen in $\mathcal{S}'$, calculate the number density n'_+ of *positrons* within the cylindrical volume. (You may use elementary arguments involving space contraction, or you may use the fact that $(c\rho, \vec{J})$ is a 4-vector, where ρ is the charge density (coul/m^3) and $\vec{J}$ is the current density (amps/m^2).)

Hint:

Considering only the positrons, write an inverse Lorentz transformation for $c\rho_+$. For this particular relative velocity between $\mathcal{S}'$ and $\mathcal{S}$, what is $\vec{J}'_+$? How is $\rho_+^{(\prime)}$ related to $n_+^{(\prime)}$?

(d.)

As seen in $\mathcal{S}'$, calculate the number density n'_- of *electrons* within the cylindrical volume.

Hint:

Considering only the electrons, write a direct Lorentz transformation for $c\rho'_-$. Because $\mathcal{S}'$ is not at rest with respect to the electrons, your result will be slightly less elementary than your result for n'_+.

(e.)

Calculate the (cylindrically radial) electric field $\vec{E}'$ seen outside the channel in $\mathcal{S}'$. Do this both

- by using the results of (c.) and (d.) plus Gauss's law, and
- by using the results of (b.) plus the rules for relativistic $\vec{E}$ and $\vec{B}$ field transformations.

Hint:

Taking advantage of the trivial value of $\vec{E}$ in $\mathcal{S}$, write a direct Lorentz transformation for $\vec{E}'_\perp$ (cf. SCSR Eq. (42)). Use $\epsilon_0\mu_0 = 1/c^2$ to check that both answers agree.

25.

(a.)

Express $\mu_0 J^\nu$ as the four-divergence of the field strength tensor $F^{\mu\nu}$. Exploiting the antisymmetry of $F^{\mu\nu}$ under interchange of its indices, prove without reference to the specific values of the elements of F that

$$\partial_\mu J^\mu = 0$$

and thus that electric charge must be conserved. (The basic structure of Maxwell's equations would have to be completely reformulated if even the tiniest violation of electric charge conservation were to be observed anywhere in the universe.)

Hint:

Consider the sum $A_{\mu\nu}B^{\mu\nu}$, where A and B are any two Lorentz four-tensors. Suppose that A is even under the interchange of μ and ν, while B is odd. Consider a particular set of values of μ and ν, for example 0 and 2, yielding a term $A_{02}B^{02}$ in the sum. However, since B but not A is odd under interchange of indices, this term will be cancelled by another term $A_{20}B^{20}$. In this argument, if B is well-behaved under differentiation, can $\partial_\mu\partial_\nu$ play the same role as $A_{\mu\nu}$?

(b.)

Define

$$\epsilon_{\mu\nu\rho\sigma} \equiv g_{\mu\alpha}g_{\nu\beta}g_{\rho\kappa}g_{\sigma\lambda}\epsilon^{\alpha\beta\kappa\lambda} \ ,$$

where ϵ is as defined after SCSR Eq. (64). Prove that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma} \ .$$

Hint:

When an index μ is changed from a superscript (contravariant) to a subscript (covariant), or *vice versa*, a sign change is required if the index is spacelike ($1 \le \mu \le 3$) but not if it is timelike ($\mu = 0$). If $\epsilon^{\mu\nu\rho\sigma}$ is nonzero, how many of its indices are spacelike?

(c.)

Without making reference to the specific values of the dual field strength tensor

$$G^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \ ,$$

using both the antisymmetry of F and the antisymmetry of ϵ, prove that

$$\partial_\mu G^{\mu\nu} = 0 \ .$$

(This is equivalent to the sourceless Maxwell equations.)

Hint:

Write out $F_{\rho\sigma}$ in terms of $\partial_{\rho,\sigma}$ and $A_{\rho,\sigma}$. Use the fact that $\epsilon^{\mu\nu\rho\sigma}$ is odd under interchange of ρ and σ to combine the two terms. Now consider $\partial_\mu G^{\mu\nu}$. Recalling the argument made in part (a.), what is the behavior of ϵ when μ and ρ are interchanged?